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Berezinian Construction of Super-Solitons in Supersymmetric Constrained KP Hierarchies

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Abstract

We consider a broad class $SKP_{\frac{r}{2},\frac{m}{2}}$ of consistently reduced Manin-Radul supersymmetric KP hierarchies (MR-SKP) which are supersymmetric analogs of the ordinary bosonic constrained KP models. Compatibility of these reductions to $SKP_{\frac{r}{2},\frac{m}{2}}$ with the MR fermionic isospectral flows is achieved via appropriate modification of the latter preserving their (anti-)commutation algebra. Unlike the general unconstrained MR-SKP case, Darboux-Bäcklund transformations do preserve the fermionic isospectral flows of $SKP_{\frac{r}{2},\frac{m}{2}}$. This allows for a systematic derivation of explicit Berezinian solutions for the $SKP_{\frac{r}{2},\frac{m}{2}}$ super-tau-functions (super-solitons).

Introduction

Manin-Radul supersymmetric KP (MR-SKP) integrable hierarchy of nonlinear evolution ("super-soliton") equations [1] and other related supersymmetric integrable hierarchies [2, 3, 4, 5, 6, 7, 8] attracted a lot of interest, both from purely mathematical point of view as supersymmetric generalizations of the inverse scattering method, bi-Hamiltonian structures, tau-functions and Sato Grassmannian approach, as well as in the context of theoretical physics due to their relevance in non-perturbative superstring theory [9].

In the present paper we will be specifically concerned with MR-SKP hierarchy [1], *i.e.*, possessing $N = 1$ supersymmetry and being defined in terms of *fermionic* (Grassmann-odd) pseudo-differential Lax operator. In ref.[10] we have already started a systematic study of MR-SKP hierarchy with particular attention being paid to the proper treatment of the fermionic MR isospectral flows, which was lacking in the previous studies on the subject. In [10] we introduced an infinite algebra of commuting additional ("ghost") symmetries of MR-SKP hierarchy which were used to construct systematic reductions to a broad class of constrained supersymmetric KP hierarchies denoted as $SKP_{\frac{r}{2},\frac{m}{2}}$ (see

Eq.(12) below; we will keep in the sequel the name MR-SKP to explicitly denote the full unconstrained hierarchy). The constrained $SKP_{\frac{r}{2},\frac{m}{2}}$ hierarchies possess correct evolution under the bosonic (Grassmann-even) isospectral flows. However, it turns out that the reductions from MR-SKP to $SKP_{\frac{r}{2},\frac{m}{2}}$ hierarchies are *incompatible* with the original MR-SKP fermionic (Grassmann-odd) isospectral flows. In [10] we provided a solution to this problem for the simplest case of constrained $SKP_{\frac{1}{2},\frac{1}{2}}$ hierarchy by appropriately modifying MR-SKP fermionic flows while preserving their original (anti-)commutation algebra, *i.e.*, preserving the integrability of the constrained $SKP_{\frac{1}{2},\frac{1}{2}}$ system. One of the results of the present paper is the extention of this construction to all $SKP_{\frac{r}{2},\frac{m}{2}}$ hierarchies.

Our next result concerns the construcion of Darboux-Bäcklund (DB) transformations preserving both types (even and odd) of the isospectral flows. As already pointed out in [10], DB transformations are *always incompatible* with the fermionic flows in the original unconstrained MR-SKP hierarchy. However, for constrained $SKP_{\frac{r}{2},\frac{m}{2}}$ hierarchies the compatibility of DB transformations is here achieved thanks to the above mentioned modification of the original MR-SKP fermionic flows.

Furthermore, we provide explicit expressions for the super-tau function and the super-eigenfunctions on DB-orbits of iterations of the DB transformations for arbitrary constrained $SKP_{\frac{r}{2},\frac{m}{2}}$ hierarchies, which are given in terms of Wronskian-like Berezinians. These Berezinian solutions constitute supersymmetric generalizations of the (multi-)soliton solutions in ordinary bosonic KP hierarchies.

Background on Manin-Radul Supersymmetric KP Hierarchy

MR-SKP hierarchy is defined through the *fermionic* $N=1$ super-pseudo-differential Lax operator $\mathcal{L}$:

$$\mathcal{L} = \mathcal{D} + f_0 + \sum_{j=1}^{\infty} b_j \partial^{-j} \mathcal{D} + \sum_{j=1}^{\infty} f_j \partial^{-j} \tag{1}$$

where the coefficients b_j, f_j are bosonic and fermionic superfield functions, respectively. We shall use throughout this paper the super-pseudo-differential calculus [1] with the following notations: ∂ and $\mathcal{D} = \frac{\partial}{\partial\theta} + \theta\partial$ denote operators, whereas the symbols ∂_x and $\mathcal{D}_\theta$ will indicate application of the corresponding operators on superfield functions. As usual, (x, θ) denote $N = 1$ superspace coordinates. For any super-pseudo-differential operator $\mathcal{A} = \sum_j a_{j/2}\mathcal{D}^j$ the subscripts $(\pm)$ denote its purely differential part $(\mathcal{A}_+ = \sum_{j\geq 0} a_{j/2}\mathcal{D}^j)$ or its purely pseudo-differential part $(\mathcal{A}_- = \sum_{j\geq 1} a_{-j/2}\mathcal{D}^{-j})$, respectively. For any $\mathcal{A}$ the super-residuum is defined as $\mathcal{R}es\mathcal{A} = a_{-\frac{1}{2}}$.

The Lax evolution Eqs. for MR-SKP read [1] :

$$\frac{\partial}{\partial t_l}\mathcal{L} = -\left[\mathcal{L}_-^{2l}, \mathcal{L}\right] = \left[\mathcal{L}_+^{2l}, \mathcal{L}\right] \tag{2}$$

$$D_n\mathcal{L} = -\left\{\mathcal{L}_-^{2n-1}, \mathcal{L}\right\} = \left\{\mathcal{L}_+^{2n-1}, \mathcal{L}\right\} - 2\mathcal{L}^{2n} \tag{3}$$

with the short-hand notations:

$$D_n = \frac{\partial}{\partial\theta_n} - \sum_{k=1}^{\infty}\theta_k \frac{\partial}{\partial t_{n+k-1}} \quad , \quad \{ D_k \, , \, D_l \} = -2\frac{\partial}{\partial t_{k+l-1}} \tag{4}$$

$$(t,\theta) \equiv (t_1 \equiv x, t_2, \ldots ; \theta, \theta_1, \theta_2, \ldots) \tag{5}$$

An important rôle in the present approach is played by the notion of (adjoint-) super-eigenfunctions (sEF's) $\Phi = \Phi(t,\theta)$ and $\Psi = \Psi(t,\theta)$ of MR-SKP hierarchy obeing:

$$\frac{\partial}{\partial t_l}\Phi = \mathcal{L}^{2l}_{+}(\Phi) \quad , \quad D_n\Phi = \mathcal{L}^{2n-1}_{+}(\Phi)$$

$$\frac{\partial}{\partial t_l}\Psi = -\left(\mathcal{L}^{2l}\right)^{*}_{+}(\Psi) \quad , \quad D_n\Psi = -\left(\mathcal{L}^{2n-1}\right)^{*}_{+}(\Psi) \tag{6}$$

The (adjoint-)super-Baker-Akhiezer functions $\psi^{(*)}_{BA}$ of MR-SKP are particular cases of (adjoint-)sEF's which satisfy the spectral equations $\left(\mathcal{L}^2\right)^{(*)}\psi^{(*)}_{BA} = \pm\lambda\psi^{(*)}_{BA}$ in addition to (6).

Finally, the super-tau-function $\tau(t,\theta)$ is expressed in terms of the super-residues of powers of the super-Lax operator (1) as follows:

$$\mathcal{R}es\mathcal{L}^{2k} = \frac{\partial}{\partial t_k}\mathcal{D}_\theta \ln\tau \quad , \quad \mathcal{R}es\mathcal{L}^{2k-1} = D_k\mathcal{D}_\theta \ln\tau \tag{7}$$

Constrained Supersymmetric KP Hierarchies

Let us consider an infinite set $\left\{\Phi_{j/2}, \Psi_{j/2}\right\}_{j=0}^{\infty}$ of pairs of (adjoint-)sEF's of $\mathcal{L}$ where j indicates their Grassmann parity (integer indices correspond to bosonic, whereas half-integer indices correspond to fermionic parity). It was shown in [10] that the following infinite set of super-pseudo-differential operators:

$$\mathcal{M}_{s/2} = \sum_{k=0}^{s-1}\Phi_{\frac{s-1-k}{2}}\mathcal{D}^{-1}\Psi_{\frac{k}{2}} \quad , \quad s = 1,2,\ldots \tag{8}$$

generate an infinite set of flows $\bar{\partial}_{s/2}$ ($\bar{\partial}_{n-\frac{1}{2}} \equiv \bar{D}_n \;\; , \;\; \bar{\partial}_k \equiv \frac{\partial}{\partial \bar{t}_k}$) :

$$\bar{D}_n\mathcal{L} = \left\{ \mathcal{M}_{n-\frac{1}{2}} \, , \, \mathcal{L} \right\} \quad , \quad \frac{\partial}{\partial \bar{t}_k}\mathcal{L} = \left[\mathcal{M}_k \, , \, \mathcal{L} \right] \tag{9}$$

which (anti-)commute with the original isospectral flows $\frac{\partial}{\partial t_l}, D_n$ (2)–(3), *i.e.*, $\bar{\partial}_{s/2}$ define an infinite-dimensional algebra of additional "ghost" symmetries of MR-SKP hierarchy, obeying the (anti-)commutation relations:

$$\left[\frac{\partial}{\partial \bar{t}_s} \, , \, \frac{\partial}{\partial \bar{t}_k} \right] = \left[\frac{\partial}{\partial \bar{t}_s} \, , \, \bar{D}_n \right] = 0 \quad , \quad \left\{ \bar{D}_i \, , \, \bar{D}_j \right\} = -2\frac{\partial}{\partial \bar{t}_{i+j-1}} \tag{10}$$

The super-"ghost"-symmetry flows and the corresponding generating operators $\mathcal{M}_{\frac{s}{2}}$ (8)–(9) are used to construct a series of reductions of the MR-SKP hierarchy [10]. Since the super-"ghost" flows obey the same algebra (10) as the original MR-SKP flows (4), one can identify an infinite subset of the latter with a corresponding infinite subset of the former:

$$\partial_{\ell \frac{r}{2}} = -\bar{\partial}_{\ell \frac{m}{2}} \quad , \quad \ell = 1, 2, \ldots \quad ; \quad \partial_k \equiv \frac{\partial}{\partial t_k} \; , \; \partial_{k-\frac{1}{2}} \equiv D_k \quad ; \quad \bar{\partial}_k \equiv \frac{\partial}{\partial \bar{t}_k} \; , \; \bar{\partial}_{k-\frac{1}{2}} \equiv \bar{D}_k \tag{11}$$

where (r, m) are some fixed positive integers of *equal parity*, and retain only these flows as Lax evolution flows (this is a supersymmetric extension of the usual reduction procedure in the purely bosonic case [11]). Eqs.(11) imply the identification $\left(\mathcal{L}^{r\ell}\right)_- = \mathcal{M}_{\ell \frac{m}{2}}$ for any ℓ. Therefore, the pertinent reduced (constrained) MR-SKP hierarchy, denoted as $SKP_{\frac{r}{2}, \frac{m}{2}}$, is described by the following constrained super-Lax operator:

$$\mathcal{L}_{(\frac{r}{2}, \frac{m}{2})} = \mathcal{D}^r + \sum_{i=0}^{r-1} v^{(r)}_{\frac{i}{2}} \mathcal{D}^i + \sum_{j=0}^{m-1} \Phi_{\frac{m-1-j}{2}} \mathcal{D}^{-1} \Psi_{\frac{j}{2}} \tag{12}$$

which is the supersymmetric counterpart of the ordinary pseudo-differential Lax operator describing the bosonic constrained KP hierarchies $\mathsf{cKP}_{r,m}$ (for a detailed discussion and further references, see [12]).

Henceforth we will restrict our attention to *fermionic* constrained $SKP_{\frac{r}{2}, \frac{m}{2}}$ hierarchies, *i.e.*, (12) with (r, m) being odd integers.

As already pointed out in [10], the original MR fermionic flows (3) are incompatible with the reduction of MR-SKP (1) to fermionic constrained $SKP_{\frac{r}{2}, \frac{m}{2}}$ hierarchies (12). Namely, taking the $(-)$ part of Eqs.(3) for fermionic constrained $\mathcal{L}_{(\frac{r}{2}, \frac{m}{2})}$ (Eq.(12) with $r, m = odd$) and using a series of simple identities for super-pseudo-differential operators [10] we obtain:

$$\sum_{j=0}^{m-1} \left[\Big(D_n \Phi_{\frac{m-1-j}{2}} - \mathcal{L}^{2n-1}_+ (\Phi_{\frac{m-1-j}{2}}) \Big) \mathcal{D}^{-1} \Psi_{\frac{j}{2}} \, \Phi_{\frac{m-1-j}{2}} \mathcal{D}^{-1} \Big(D_n \Psi_{\frac{j}{2}} + \left(\mathcal{L}^{2n-1}\right)^*_+ (\Psi_{\frac{j}{2}}) \Big) \right]$$

$$= -2 \sum_{j=0}^{m-1} \sum_{k=0}^{2n-1} \mathcal{L}^{2n-1-k} (\Phi_{\frac{m-1-j}{2}}) \mathcal{D}^{-1} \mathcal{L}^{k^*} (\Psi_{\frac{j}{2}}) \tag{13}$$

which leads to apparent contradiction, since the l.h.s. of (13) vanishes by virtue of Eqs.(6) for the (adjoint-)sEF's, whereas the r.h.s. of (13) is manifestly non-zero.

Generalizing the argument given in [10] for the simplest $SKP_{\frac{1}{2}, \frac{1}{2}}$ case, we arrive at the following:

Proposition 1 *There exists the following consistent modification of MR-SKP flows D_n (3) for constrained $SKP_{\frac{r}{2}, \frac{m}{2}}$ hierarchy $(r, m = odd)$:*

$$\mathcal{D}_k \mathcal{L} = - \Big\{ \mathcal{L}^{2k-1}_- - X^{(2k-1)} \, , \, \mathcal{L} \Big\} = \big\{ \mathcal{L}^{2k-1}_+ \, , \, \mathcal{L} \big\} + \Big\{ X^{(2k-1)} \, , \, \mathcal{L} \Big\} - 2\mathcal{L}^{2k} \tag{14}$$

$$X^{(2k-1)} \equiv 2 \sum_{j=0}^{m-1} \sum_{l=0}^{k-2} \mathcal{L}^{2(k-l)-3} (\Phi_{\frac{m-j-1}{2}}) \mathcal{D}^{-1} \left(\mathcal{L}^{2l+1}\right)^* (\Psi_{\frac{j}{2}}) \tag{15}$$

$$\mathcal{D}_k \Phi_{\frac{j}{2}} = \mathcal{L}_+^{2k-1}(\Phi_{\frac{j}{2}}) - 2\mathcal{L}^{2k-1}(\Phi_{\frac{j}{2}}) + X^{(2k-1)}(\Phi_{\frac{j}{2}}) \tag{16}$$

$$\mathcal{D}_k \Psi_{\frac{j}{2}} = -\left(\mathcal{L}^{2k-1}\right)_+^* (\Psi_{\frac{j}{2}}) + 2\left(\mathcal{L}^{2k-1}\right)^* (\Psi_{\frac{j}{2}}) - \left(X^{(2k-1)}\right)^* (\Psi_{\frac{j}{2}}) \tag{17}$$

The modified $\mathcal{D}_k$ flows for $SKP_{\frac{r}{2},\frac{m}{2}}$ obey the same anti-commutation algebra $\{\mathcal{D}_k,\, \mathcal{D}_l\} = -2\frac{\partial}{\partial t_{r(k+l-1)}}$ as in the original unconstrained case (4) (modulo r).

Remark. For bosonic $SKP_{\frac{r}{2},\frac{m}{2}}$ models (Eq.(12) with $r, m = even$) there is no need to modify MR fermionic flows, since in this case the term in r.h.s. of (13) is absent.

Berezinian Solutions for the Super-Tau Function

It was demostrated in [10] that for the general MR-SKP hierarchy (1) the Darboux-Bäcklund (DB) transformations $\widetilde{\mathcal{L}} = \mathcal{T}\mathcal{L}\mathcal{T}^{-1}$, where $\mathcal{T} = \chi\mathcal{D}\chi^{-1}$ with χ being a bosonic sEF (6) of $\mathcal{L}$, do *not* preserve the fermionic-flow Lax Eqs.(3). Indeed, for the DB-transformed $\widetilde{\mathcal{L}}$ to obey the same MR flow Eqs.(2)–(3) as $\mathcal{L}$, the DB-generating "gauge" transformation $\mathcal{T}$ must satisfy:

$$\frac{\partial}{\partial t_l}\mathcal{T}\,\mathcal{T}^{-1} + \left(\mathcal{T}\mathcal{L}_+^{2l}\mathcal{T}^{-1}\right)_- = 0 \quad , \quad D_n\mathcal{T}\,\mathcal{T}^{-1} - \left(\mathcal{T}\mathcal{L}_+^{2n-1}\mathcal{T}^{-1}\right)_- = -2\left(\widetilde{\mathcal{L}}^{2n-1}\right)_- \tag{18}$$

The first Eq.(18) is exactly analogous to the purely bosonic case and implies that χ must be a sEF (6) of $\mathcal{L}$ w.r.t. the even MR-SKP flows. However,the second Eq.(18) does not have solutions for χ for the general MR-SKP hierarchy. In particular, if χ would be a sEF also w.r.t. fermionic flows (cf. second Eq.(6)), then the l.h.s. of second Eq.(18) would become zero thereby leading to the contradictory relation: $\left(\widetilde{\mathcal{L}}^{2n-1}\right)_- = 0$. This makes the standard DB method inapplicable to find solutions of the unconstrained MR-SKP.

On the other hand, for constrained fermionic $SKP_{\frac{r}{2},\frac{m}{2}}$ hierarchies it can easily be shown (extending the proof given in [10] for the simplest $SKP_{\frac{1}{2},\frac{1}{2}}$ case), that *auto*-DB transformations (*i.e.*, those preserving the constrained form (12) of the initial $SKP_{\frac{r}{2},\frac{m}{2}}$ hierarchy) are compatible with the *modified* fermionic flows (14)–(17). This latter result guarantees that any iteration of DB transformations of the initial $SKP_{\frac{r}{2},\frac{m}{2}}$ hierarchy (in particular, the "free" one with $\mathcal{L}^{(0)}_{(\frac{r}{2},\frac{m}{2})} = \mathcal{D}^r$) will yield new nontrivial solutions for $SKP_{\frac{r}{2},\frac{m}{2}}$, which obey the same isospectral flow Eqs.(2),(14),(16)–(17), *i.e.*, both bosonic *and* fermionic, as the initial hierarchy.

Now, consider auto-DB transformations for arbitrary $SKP_{\frac{r}{2},\frac{m}{2}}$ (12) (here $L \equiv \mathcal{L}_{(\frac{r}{2},\frac{m}{2})} \equiv \mathcal{L}^{(0)}_{(\frac{r}{2},\frac{m}{2})}$) :

$$\widetilde{L} = \mathcal{T}_a L \mathcal{T}_a^{-1} = \widetilde{L}_+ + \sum_{j=0}^{m-1} \widetilde{\Phi}_{\frac{m-j-1}{2}} \mathcal{D}^{-1} \widetilde{\Psi}_{j/2} \tag{19}$$

$$\widetilde{\Phi}_a = \mathcal{T}_a L(\Phi_a) \quad , \quad \widetilde{\Psi}_a = \Phi^{-1}_{\frac{m-2a-1}{2}}, \quad \widetilde{\Phi}_{\frac{m-j-1}{2}} = \mathcal{T}_a(\Phi_{\frac{m-j-1}{2}})$$

$$\widetilde{\Psi}_{j/2} = (-1)^{j+1}\mathcal{T}_a^{*-1}(\Psi_{j/2}) = (-1)^j \Phi_a^{-1}\mathcal{D}_\theta^{-1}(\Phi_a \Psi_{\frac{j}{2}}) \quad \text{for } j \neq m-2a-1 \tag{20}$$

where $\mathcal{T}_a = \Phi_a \mathcal{D} \Phi_a^{-1}$ with a being a fixed integer (bosonic) index. Under DB transformations the super-tau function transforms as (cf. Eq.(3.4) in [10]) :

$$\widetilde{\tau} = \Phi_a \tau^{-1} \tag{21}$$

Before proceeding to the iteration of DB-transformations for $SKP_{\frac{r}{2},\frac{m}{2}}$ hierarchies (19)–(20), we will introduce some convenient short-hand notations for Wronskian-type Berezinians:

$$\mathrm{Ber}_{(k,l)}[\varphi_0,\ldots,\varphi_{k-1};\varphi_{\frac{1}{2}},\ldots,\varphi_{l-\frac{1}{2}}] \equiv \mathrm{Ber}\begin{pmatrix} \mathcal{W}_{k,k}[\varphi_0,\ldots,\varphi_{k-1}] & | & \mathcal{W}_{k,l}[\varphi_{\frac{1}{2}},\ldots,\varphi_{l-\frac{1}{2}}] \\ ------------- & | & ------------- \\ \mathcal{W}_{l,k}[\mathcal{D}_\theta\varphi_0,\ldots,\mathcal{D}_\theta\varphi_{k-1}] & | & \mathcal{W}_{l,l}[\mathcal{D}_\theta\varphi_{\frac{1}{2}},\ldots,\mathcal{D}_\theta\varphi_{l-\frac{1}{2}}] \end{pmatrix} \tag{22}$$

where $(\varphi_0,\ldots,\varphi_{k-1})$ and $\left(\varphi_{\frac{1}{2}},\ldots,\varphi_{l-\frac{1}{2}}\right)$ are sets of bosonic (fermionic) superfield functions, and where $\mathcal{W}_{k,l}[f_1,\ldots,f_l]$ denotes a *rectangular* (k rows by l columns) Wronskian matrix:

$$\mathcal{W}_{k,l}[f_1,\ldots,f_l] = \left\|\partial_x^{\alpha-1} f_\beta\right\| \quad , \quad \alpha=1,\ldots,k \ , \ \beta=1,\ldots,l \tag{23}$$

The derivation of the explicit form of the DB-orbit for the super-tau function and the (adjoint-)EF's of $SKP_{\frac{r}{2},\frac{m}{2}}$ is based on the following Proposition:

Proposition 2 *The iteration of Darboux-Bäcklund -like transformations on arbitrary initial superfield functions (Φ – bosonic, F – fermionic) can be expressed in a Berezinian form as follows:*

$$\Phi^{(2n)} \equiv \mathcal{T}^{(2n-1)}_{\varphi_{n-\frac{1}{2}}} \mathcal{T}^{(2n-2)}_{\varphi_{n-1}} \ldots \mathcal{T}^{(3)}_{\varphi_{3/2}} \mathcal{T}^{(2)}_{\varphi_1} \mathcal{T}^{(1)}_{\varphi_{\frac{1}{2}}} \mathcal{T}^{(0)}_{\varphi_0}(\Phi) = \tag{24}$$

$$\left(\mathrm{Ber}_{(n,n)}[\varphi_0,\ldots,\varphi_{n-1},\Phi;\varphi_{\frac{1}{2}},\ldots,\varphi_{n-\frac{1}{2}}]\right)^{-1} \mathrm{Ber}_{(n+1,n)}[\varphi_0,\ldots,\varphi_{n-1},\Phi;\varphi_{\frac{1}{2}},\ldots,\varphi_{n-\frac{1}{2}}]$$

$$F^{(2n+1)} \equiv \mathcal{T}^{(2n)}_{\varphi_n} \mathcal{T}^{(2n-1)}_{\varphi_{n-\frac{1}{2}}} \ldots \mathcal{T}^{(3)}_{\varphi_{3/2}} \mathcal{T}^{(2)}_{\varphi_1} \mathcal{T}^{(1)}_{\varphi_{\frac{1}{2}}} \mathcal{T}^{(0)}_{\varphi_0}(F) = \tag{25}$$

$$\mathrm{Ber}_{(n+1,n)}[\varphi_0,\ldots,\varphi_n;\varphi_{\frac{1}{2}},\ldots,\varphi_{n-\frac{1}{2}}] \left(\mathrm{Ber}_{(n+1,n+1)}[\varphi_0,\ldots,\varphi_n;\varphi_{\frac{1}{2}},\ldots,\varphi_{n-\frac{1}{2}},F]\right)^{-1}$$

where by definition:

$$\mathcal{T}^{(j)}_{\varphi_{\frac{j}{2}}} = \varphi^{(j)}_{\frac{j}{2}} \mathcal{D} \frac{1}{\varphi^{(j)}_{\frac{j}{2}}} \quad , \quad \varphi^{(j)}_{\frac{j}{2}} = \mathcal{T}^{(j-1)}_{\varphi_{\frac{j-1}{2}}} \mathcal{T}^{(j-2)}_{\varphi_{\frac{j}{2}-1}} \ldots \mathcal{T}^{(2)}_{\varphi_1} \mathcal{T}^{(1)}_{\varphi_{\frac{1}{2}}} \mathcal{T}^{(0)}_{\varphi_0}(\varphi_{\frac{j}{2}}) \tag{26}$$

Here and in what follows the superscripts in brackets indicate the step of iteration of DB(-like) transformations. Note that $F^{(2n+1)}$ (25) and $\varphi^{(2k+1)}_{k+\frac{1}{2}}$ (26) are bosonic although the initial $F, \varphi_{k+\frac{1}{2}}$ are fermionic.

The proof of Prop.2 relies on the observation, that both sides of (24) and (25) define monic super-differential operators acting on Φ and F, respectively, which share the same set of kernel elements, namely, the superfield functions $\varphi_0,\ldots,\varphi_{n-1};\varphi_{\frac{1}{2}},\ldots,\varphi_{n-\frac{1}{2}}$.

Let us consider in more detail the DB-orbit of constrained $SKP_{\frac{r}{2},\frac{m}{2}}$ hierarchy with $r = 1$, *i.e.*, $L \equiv \mathcal{L}_{(\frac{1}{2},\frac{m}{2})} = \mathcal{D} + f_0 + \sum_{j=0}^{m-1} \Phi_{\frac{m-1-j}{2}} \mathcal{D}^{-1} \Psi_{\frac{j}{2}}$ (in the formulas below m indicates the order of the pseudo-differential part of $L \equiv \mathcal{L}_{(\frac{1}{2},\frac{m}{2})}$, the integer k is $0 \leq k \leq m-1$, and l is arbitrary non-negative integer) :

$$\Phi^{(ml+k)} = \left(\mathcal{T}^{(lm-1+k)}_{\Phi_{\frac{k-1}{2}}} \ldots \mathcal{T}^{(lm)}_{\Phi_0}\right) \cdots \left(\mathcal{T}^{(m-1)}_{\Phi_{\frac{m-1}{2}}} \ldots \mathcal{T}^{(0)}_{\Phi_0}\right) \left(L^{l+1}(\Phi_{\frac{j}{2}})\right) \quad \text{for } 0 \leq j \leq k-1 \tag{27}$$

$$\Phi^{(ml+k)} = \left(\mathcal{T}^{(lm-1+k)}_{\Phi_{\frac{k-1}{2}}} \ldots \mathcal{T}^{(lm)}_{\Phi_0}\right) \cdots \left(\mathcal{T}^{(m-1)}_{\Phi_{\frac{m-1}{2}}} \ldots \mathcal{T}^{(0)}_{\Phi_0}\right) \left(L^{l}(\Phi_{\frac{j}{2}})\right) \quad \text{for } k \leq j \leq m-1 \tag{28}$$

Eqs.(27)–(28) indicate that the DB-orbit is defined by successive iterations of DB-transformations w.r.t. all super-EF's $\Phi_{\frac{j}{2}}$ $(j = 0, \ldots, m-1)$ present in $L \equiv \mathcal{L}_{(\frac{1}{2},\frac{m}{2})}$. Comparing (27)–(28) with the general formulas (24)–(25) we easily identify the functions φ_k and $\varphi_{\frac{k}{2}}$ appearing in the latter with the super-EF's $\Phi_{\frac{j}{2}}$ of $L \equiv \mathcal{L}_{(\frac{1}{2},\frac{m}{2})}$ as follows:

$$\varphi_{\frac{ml+j}{2}} = L^l(\Phi_{\frac{j}{2}}) \tag{29}$$

Therefore, the explicit expressions for the super-tau functions on the DB-orbit (27)–(28), upon using (21) and (24)–(25), are given by:

$$\tau^{(2n+1)} = \mathrm{Ber}_{(n+1,n)}[\varphi_0, \ldots, \varphi_n; \varphi_{\frac{1}{2}}, \ldots, \varphi_{n-\frac{1}{2}}] \frac{1}{\tau^{(0)}} \tag{30}$$

$$\tau^{(2n)} = \left(\mathrm{Ber}_{(n,n)}[\varphi_0, \ldots, \varphi_{n-1}; \varphi_{\frac{1}{2}}, \ldots, \varphi_{n-\frac{1}{2}}]\right)^{-1} \tau^{(0)} \tag{31}$$

with the substitution (29) for $\varphi_k, \varphi_{\frac{k}{2}}$ in the r.h.s. of (30)–(31).

Super-Soliton Solutions

Now, let us provide some explicit examples of Berezinian solutions for the $SKP_{\frac{r}{2},\frac{m}{2}}$ tau-function (30)–(31). We shall consider the simplest case of constrained $SKP_{\frac{1}{2},\frac{1}{2}}$ hierarchy and take the initial $\tau^{(0)} = const$, *i.e.*, the initial super-Lax operator being $L \equiv \mathcal{L}_{(\frac{1}{2},\frac{1}{2})} = \mathcal{D}$. The initial super-EF $\Phi_0 \equiv \Phi_0^{(0)}$ satisfies according to (16) :

$$\frac{\partial}{\partial t_k}\Phi_0 = \partial_x^k \Phi_0 \quad , \quad \mathcal{D}_n \Phi_0 = -\mathcal{D}_\theta^{2n-1} \Phi_0 \tag{32}$$

$$\Phi_0(t,\theta) = \int d\lambda \left[\varphi_B(\lambda) + \left(\theta - \sum_{n\geq 1} \lambda^{n-1}\theta_n\right)\varphi_F(\lambda)\right] e^{\sum_{l\geq 1} \lambda^l (t_l + \theta\theta_l)} \tag{33}$$

where $\varphi_B(\lambda)$, $\varphi_F(\lambda)$ are arbitrary bosonic (fermionic) "spectral" densities.

It is easy to show that for $SKP_{\frac{1}{2},\frac{1}{2}}$ case the Berezinian expressions (30)–(31), together with the substitution (29), which now ($m = 1$, $j = 0$) becomes simply $\varphi_{\frac{l}{2}} = \mathcal{D}^l \Phi_0$, reduce to the following ratios of ordinary Wronskians:

$$\tau^{(2n)} = \frac{W_n \left[\partial_x \Phi_0, \ldots, \partial_x^n \Phi_0\right]}{W_n \left[\Phi_0, \ldots, \partial_x^{(n-1)} \Phi_0\right]} \quad , \quad \tau^{(2n+1)} = \frac{W_{n+1} \left[\Phi_0, \ldots, \partial_x^n \Phi_0\right]}{W_n \left[\partial_x \Phi_0, \ldots, \partial_x^n \Phi_0\right]} \tag{34}$$

where Φ_0 is given by (33). In particular, choosing for the bosonic (fermionic) "spectral" densities in Eq.(33) $\varphi_B(\lambda) = \sum_{i=1}^N c_i \delta(\lambda - \lambda_i)$, $\varphi_F(\lambda) = \sum_{i=1}^N \epsilon_i \delta(\lambda - \lambda_i)$, where c_i, λ_i and ϵ_i are Grassmann-even and Grassmann-odd constants, respectively, we have for Φ_0 :

$$\Phi_0 = \sum_{i=1}^N \Big[c_i + \Big(\theta - \sum_{n \geq 1} \lambda_i^{n-1} \theta_n \Big) \Big] e^{\sum_{l \geq 1} \lambda_i^l (t_l + \theta \theta_l)} \tag{35}$$

Substituting (35) into (34) we obtain the following "super-soliton" solutions for the super-tau function:

$$\tau^{(2n+1)} = \frac{\sum_{1 \leq i_1 < \ldots < i_{n+1} \leq N} \binom{N}{n+1} \widetilde{c}_{i_1} \ldots \widetilde{c}_{i_{n+1}} E_{i_1} \ldots E_{i_{n+1}} \Delta^2_{n+1}(\lambda_{i_1}, \ldots, \lambda_{i_{n+1}})}{\sum_{1 \leq j_1 < \ldots < j_n \leq N} \binom{N}{n} \widetilde{c}_{j_1} \ldots \widetilde{c}_{j_n} E_{j_1} \ldots E_{j_n} \lambda_{j_1} \ldots \lambda_{j_n} \Delta^2_n(\lambda_{j_1}, \ldots, \lambda_{j_n})} \tag{36}$$

$$\widetilde{c}_i \equiv c_i + \Big(\theta - \sum_{n \geq 1} \lambda_i^{n-1} \theta_n \Big) \quad , \quad E_i \equiv e^{\sum_{l \geq 1} \lambda_i^l (t_l + \theta \theta_l)}$$

$$\Delta_n(\lambda_{i_1}, \ldots, \lambda_{i_n}) \equiv \det \left\| \lambda_{i_a}^{b-1} \right\|_{a,b=1,\ldots,n} \tag{37}$$

Outlook. There is a number of interesting issues, related to the present topic, which deserve further study such as: *binary* DB-transformations and new types of solutions for the super-tau-function; consistent formulation of supersymmetric two-dimensional Toda lattice and search for proper supersymmetric counterparts of random (multi-)matrix models, based on analogous approach [13] in the purely bosonic case.

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